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LINEAR FRACTAL GAMES OF PURSUIT†

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A general method for solving game problems of pursuit is proposed for dynamical systems with Volterra evolution. The method makes use of resolving functions [1] and the tools of the theory of set-valued mappings. The scheme proposed covers a wide range of functional-differential systems, such as integral, integrodifferential and differential-difference systems of equations defining the dynamics of conflict-controlled processes. A more detailed study is made of game problems for systems with Riemann–Liouville fractional derivatives and regularized Dzhrbashyan–Nersesyan derivatives ("fractal" games). Asymptotic representations of generalized Mittag-Löffler functions are used in the context of the method to establish sufficient conditions for the solvability of game problems.

The theory of differential games presents numerous fundamental techniques that can be used to establish conditions for the solvability of pursuit and evasion problems in suitable classes of strategies [2–9]. Prominent among these techniques are those developed by N. N. Krasovskii and his successors. Depending on the degree to which the players are mutually informed as to the state of the process and the controls chosen by the opponent, the mathematical tools used may differ. The method of resolving functions and its diverse modifications [1, 10] is conceptually close to Pontryagin's first direct method [3, 6]. It has recently been subject to active development and is being used to solve some very complicated game problems, such as problems of group and successive pursuit and problems with phase constraints [1, 7]. In particular, the method substantiates the classical rule of parallel pursuit in a fairly wide range of problems [11]. Its principal advantages are its universality and the possibility of obtaining effectively verifiable sufficient conditions for terminating the game.

1. FORMULATION OF THE PROBLEM, AUXILIARY RESULTS, AND SCHEME OF THE METHOD

Let R^n denote a real Euclidean n-space and $R_+ = \{t: t \geq 0\}$ the positive real line. Consider the process described by the equation

$$z(t) = g(t) + \int_0^t \Omega(t,\tau)\varphi(u(\tau), \upsilon(\tau))d\tau, \quad t \geq 0 \tag{1.1}$$

The function $g(t)$, $g: R_+ \to R^n$ is (Lebesgue-) measurable and bounded for $t > 0$; the matrix-valued function $\Omega(t, \tau)$, $t \geq \tau \geq 0$, is measurable as a function of t and summable as a function of τ for any $t \in R_+$. The control block is given by the function $\varphi(u, \upsilon)$, $\varphi: U \times V \to R^n$, which is jointly continuous in its variables on a direct product of non-empty compact sets U and V, that is, $U \in K(R^m)$, $V \in K(R^l)$. The controls of the players $u(\tau)$, u; $R_+ \to U$ and $\upsilon(\tau)$, υ: $R_+ \to V$ are measurable functions.

In addition to process (1.1), we are also given a cylindrical terminal set

$$M^* = M_0 + M \tag{1.2}$$

where M_0 is a linear subspace of R^n and $M \in K(L)$, L being the orthogonal complement of M_0 in R^n.

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Player I and II (u and υ, respectively) have opposite goals. Player I strives to steer the trajectory of process (1.1) to the set (1.2) in minimum time; player II aims to delay, as far as possible, the time at which the trajectory reaches the set M^*.

We shall take the side of player I, directing our attention to the opponent's choice of a control – an arbitrary measurable function with values in U. In turn, we shall assume that if the game (1.1), (1.2) is taking place in an interval $[0, T]$, then player I's control is measurable function

$$u(t) = u(g(T), \upsilon_t(\cdot)), \quad t \in [0, T], \quad u(t) \in U \tag{1.3}$$

where $\upsilon_t(\cdot) = \{\upsilon(s): 0 \le s \le t\}$ is the prehistory of the player II's control up to time t.

The aim of this paper is, considering the process (1.1), (1.2) under an informativeness condition of type (1.3), to establish sufficient conditions for the problem to be solvable in player I's favour in a certain guaranteed time, to estimate that time, and also to find controls for player I that achieve this result.

We will now describe a method for solving this problem.

Let π denote an orthogonal projection from R^n into L. Setting

$$\varphi(U, \upsilon) = \{\varphi(u, \upsilon): u \in U\}$$

we consider set-valued mappings

$$W(t, \tau, \upsilon) = \pi\Omega(t, \tau)\varphi(U, \upsilon), \quad W(t, \tau) = \bigcap_{\upsilon \in V} W(t, \tau, \upsilon)$$

on the set $\Delta \times V$ and Δ, respectively, where

$$\Delta = \{(t, \tau): 0 \le \tau \le t < \infty\}$$

Pontryagin's condition. The set-valued mapping $W(t, \tau)$ takes non-empty values on the set Δ.

By the continuity of the function $\varphi(u, \upsilon)$ and the condition $U \in K(R^m)$, the mapping $\varphi(U, \upsilon)$ is continuous with respect to υ in the Hausdorff metric. In view of the assumption regarding the matrix-valued function $\Omega(t, \tau)$, we may conclude [12] that the set-valued mappings $W(t, \tau, \upsilon)$ and $W(t, \tau)$ are measurable functions of τ.

Let $P(R^n)$ denote the totality of non-empty closed sets in the space R^n. Then, obviously,

$$W(t, \tau, \upsilon): \Delta \times V \to P(R^n), \quad W(t, \tau): \Delta \to P(R^n)$$

In that case we shall say that the set-valued mappings $W(t, \tau, \upsilon)$ and $W(t, \tau)$ are normal with respect to τ [12].

It follows from Pontryagin's conditions and the results of [1, 12] that, for any $t \ge 0$, at least one selector $\gamma(t, \tau) \in W(t, \tau)$ exists which is a measurable function of τ. By our assumptions concerning the parameters of the process (1.1), the selector $\gamma(t, \tau)$ is summable as a function of τ, $\tau \in [0, t]$, for any fixed $t \ge 0$. Put

$$\xi(t, g(t), \gamma(t, \cdot)) = \pi g(t) + \int_0^t \gamma(t, \tau) d\tau$$

where $\gamma(t, \tau)$ is the above-mentioned selector.

Using the function $\xi(t, g(t), \gamma(t, \cdot))$, we define a function

$$\alpha(t, \tau, \upsilon) = \sup\{\alpha \ge 0: [W(t, \tau, \upsilon) - \gamma(t, \tau)] \cap \alpha[M - \xi(t, g(t), \gamma(t, \cdot))] \ne \varnothing\} \tag{1.4}$$

which we call a resolving function [1]. This function will play a key role in what follows.

By virtue of our assumptions regarding the parameters of the process (1.1), as well as the results of [1], the function (1.4) is measurable with respect to τ and upper semicontinuous with respect to υ.

In what follows we shall be interested in the behaviour of $\alpha(t, \tau, \upsilon)$ as a function of the variables (τ, υ). We therefore fix t and set $\alpha(\tau, \upsilon) = \alpha(t, \tau, \upsilon)$. We shall say that the function $\alpha: [0, T] \times V \to R_+$ is superpositionally measurable if, for any measurable function $\upsilon(\tau)$, $\upsilon: [0, T] \to V$, the superposition $\alpha(\tau, \upsilon(\tau))$, $\alpha: [0, T] \to R_+$ is a measurable function of τ.

A sufficiently general assumption, guaranteeing that the function $\alpha(\tau, \upsilon)$ will be superpositionally measurable, is that the function be $(L \times B)$-measurable [13], that is, measurable relative to the α-algebra defined as the product of the σ-algebras $L[0, T]$ and $B(R^n)$. The elements of this σ-algebra are the subsets of the set $[0, T] \times R^n$ generated by sets of the form $X \times Y$, where X is a Lebesgue-measurable subset of the interval $[0, T]$ and Y is a Borel-measurable subset of R^n.

We put

$$W(T, \tau, \upsilon) - \gamma(T, \tau) = H(\tau, \upsilon), \quad M - \xi(T, g(T), \gamma(T, \cdot)) = M_1$$

and introduce a set-valued mapping

$$\mathfrak{U}(\tau, \upsilon) = \{\alpha \in R_+ : H(\tau, \upsilon) \cap \alpha M_1 \neq \varnothing\} \tag{1.5}$$

Then

$$\mathfrak{U}(\omega) = \{x \in F(\omega) : H(\omega) \cap M(\omega, x) \neq \varnothing\}$$

We will investigate the properties of set-valued mapping of the form (1.5).

The following general result generalizes a well-known proposition [12].

Lemma 1. Let $X \in P(R^k)$; let $F(\omega), F: X \to P(R^k), H(\omega), H: X \to P(R^n)$ be normal set-valued mappings, and $M(\omega, x)$, $M: X \times R^k \to P(R^n)$ a Carathéodory mapping (i.e. measurable with respect to ω and continuous with respect to x). Then the mapping

$$\mathfrak{U}(\omega) = \{x \in F(\omega) : H(\omega) \cap M(\omega, x) \neq \varnothing\}$$

is normal.

Putting $\omega = (\tau, \upsilon), x = \alpha$ in Lemma 1 and, respectively, $F(\omega) = R_+$ and $M(\omega, x) = \alpha M$, we infer that the mapping $\mathfrak{U}(\tau, \upsilon)$ will be $(L \times B)$-measurable, since the mapping $H(\tau, \upsilon)$ is $(L \times B)$-measurable because it is Lebesgue-measurable with respect to τ and continuous in υ.

We will now show that the function $\alpha(\tau, \upsilon)$ is $(L \times B)$-measurable. Indeed, since

$$\alpha(\tau, \upsilon) = \sup_{\alpha \in \mathfrak{U}(\tau, \upsilon)} \alpha = C(\mathfrak{U}(\tau, \upsilon); 1)$$

where $C(X; p)$ is the supporting function of X in the direction p, the fact that the function is $(L \times B)$-measurable follows from the $(L \times B)$-measurability of the set-valued mapping $\mathfrak{U}(\tau, \upsilon)$ [12].

Thus, the function $\alpha(\tau, \upsilon)$ is $(L \times B)$-measurable, bounded away from zero and upper semicontinuous with respect to υ. It can be shown that $\inf_{\upsilon \in V} \alpha(\tau, \upsilon)$ is a measurable function.

The following corollary of (1.4) deserves mention. If t exists for which $\xi(t, g(t), \gamma(t, \cdot)) \in M$, then $\alpha(t, \tau, \upsilon) = \infty$ for all $\tau \in [0, t]$, $\upsilon \in V$.

Define a mapping

$$T(g(\cdot), \gamma(\cdot, \cdot)) = \left\{ t \geq 0 : \int_0^t \inf_{\upsilon \in V} \alpha(t, \tau, \upsilon) d\tau \geq 1 \right\} \tag{1.6}$$

If some t exists such that the integral in (1.1) becomes $+\infty$, the inequality holds automatically. But if the inequality in (1.6) does not hold for any t, we put $T(g(\cdot), \gamma(\cdot, \cdot) = \varnothing$.

We may thus formulate our main result.

Theorem 1. Suppose the game (1.1), (1.2) satisfies Pontryagin's condition and that $M = \text{co}M$, on the assumption that, for some measurable and almost everywhere bounded mapping $g(t)$ and τ-measurable selector $\gamma(t, \tau)$, $t \geq \tau \geq 0$ of a set-valued mapping $W(t, \tau)$, it is true that $T(g(\cdot), \gamma(\cdot, \cdot)) \neq \varnothing$ and $T \in T(g(\cdot), \gamma(\cdot, \cdot), T < +\infty$. Then the trajectory of the process (1.1) can be steered to the terminal set at time T using a control of the form (1.3).

Proof. We first consider the case $\xi(T, g(T), \gamma(T, \cdot)) \equiv M$. Let $\upsilon_T(\cdot)$ be an arbitrary measurable function with values in V. Proceeding as in the approach described in [1, 10], we define a control function

$$h(t) = 1 - \int_0^t \alpha(T, \tau, \upsilon(\tau)) d\tau, \quad t \in [0, T]$$

Since the function $\alpha(T, \tau, \upsilon)$ is $(L \times B)$-measurable, it is superpositionally measurable, that is, the function $\alpha(T, \tau, \upsilon(\tau))$ is measurable. On the other hand, by our assumptions concerning the parameters of the process (1.1), (1.2), it is bounded for almost all $\tau < T$ and hence integrable over any finite time interval. It follows that $h(t)$ is a continuous, non-increasing function and $h(0) = 1$. Therefore, by our constructions, a time $t_* = t(\upsilon(\cdot))$, $t_* \in (0, T]$ exist such that $h(t_*) = 0$.

We will refer henceforth to the subintervals $[0, t_*)$ and $[t_*, T]$ as "active" and "passive", respectively. We can describe player I's method of control in each subinterval. To that end, consider the set-valued mapping

$$U(\tau, v) = \{u \in U\colon \pi\Omega(T, \tau)\varphi(u, v) - \gamma(T, \tau) \in \alpha(T, \tau, v)[M - \xi(T, g(T), \gamma(T, \cdot))]\} \qquad (1.7)$$

Since the function $\alpha(T, \tau, \upsilon)$ is $(L \times B)$-measurable, $M \in K(R^n)$ and the function $\xi(T, g(T), \gamma(T, \cdot))$ is bounded, it follows that the mapping $\alpha(T, \tau, \upsilon)[M - \xi(T, g(T), \gamma(T, \cdot))]$ is $(L \times B)$-measurable. In addition, the left-hand side of the inclusion relation in (1.7) is obviously an $(L \times B)$-measurable function of (τ, υ) and a continuous function of u. Hence, by a well-known proposition [12], it follows that the mapping $U(\tau, \upsilon)$ is $(L \times B)$-measurable. Consequently, its selector

$$u(\tau, v) = \operatorname{lex\,min} U(\tau, v) \qquad (1.8)$$

is an $(L \times B)$-measurable function. Player I's control in the active subinterval $[0, t_*)$ may now be defined as

$$u(\tau) = u(\tau, v(\tau)) \qquad (1.9)$$

Since it is $(L \times B)$-measurable, the function $u(\tau, \upsilon)$ is superpositionally measurable, and consequently $u(\tau)$ is a measurable function.

Now consider the passive subinterval $[t_*, T]$. Putting $\alpha(T, \tau, \upsilon) \equiv 0$ for $\tau \in [t_*, T]$, $\upsilon \in V$, in expression (1.7), we choose player I's control according to the previously proposed procedure, using relations (1.7)–(1.9)

In the case when $\xi(T, g(T), \gamma(T, \cdot)) \in M$, player I's control in the interval $[0, T]$ is chosen by the same considerations as in the passive subinterval, that is, in accordance with the scheme (1.7)–(1.9) with $\alpha(T, \tau, \upsilon) \equiv 0$, $\tau \in [0, T]$, $\upsilon \in V$.

We claim that, if player I's control is defined as (1.9), with allowance for (1.7) and (1.8), then in each case the trajectory of the process (1.1) will be steered at time T to the set M^*, whatever player II's controls.

It follows from (1.1) that

$$\pi z(T) = \pi g(T) + \int_0^T \pi\Omega(T, \tau)\varphi(u(\tau), v(\tau))d\tau \qquad (1.10)$$

We will first analyse the case when $\xi(T, g(t), \gamma(T, \cdot)) \notin M$. To that end we add and subtract the vector $\int_0^T \gamma(T, \tau)d\tau$ on the right of Eq. (1.10). Using the law described above to select player I's control, we infer from (1.10) that

$$\pi z(T) \in \xi(T, g(T), \gamma(T, \cdot))\left[1 - \int_0^{t_*} \alpha(T, \tau, v(\tau))d\tau\right] + \int_0^{t_*} \alpha(T, \tau, v(\tau))Md\tau \qquad (1.11)$$

Since M is convex and compact, $\alpha(T, \tau, \upsilon(\tau))$ is a non-negative function for $\tau \in [0, t_*)$ and the bracketed integral equals 1, it follows that the last integral in inclusion relation (1.11) equals M, and so $\pi z(T) \in M$, or $z(T) \in M^*$.

Let $\xi(T, g(T), \gamma(T, \cdot)) \in M$. Then, in accordance with player I's control law as specified, the inclusion $\pi z(T) \in M$ immediately follows from (1.10).

2. SYSTEMS WITH FRACTIONAL DERIVATIVES

In this section, standard techniques will be used to introduce the classical concepts of fractional integral and fractional derivative (*FD*) (in the Riemann–Liouville sense). Correspondingly, one will then have an equation with fractional derivatives (fractional-differential equation), in which the usual Cauchy data at the starting time $t = 0$ must be replaced by a fractional integral of suitable fractional order. This is because, generally speaking, the solution of such an equation has a singularity at $t = 0$, and only such generalized initial data are natural in this case. However, for physical reasons, it is desirable to have ordinary Cauchy problems for fractional-differential equations. Dzhrbashyan and Nersesyan have proposed a fractional-differential equation in which the Riemann–Liouville derivative are replaced by their regularized values and the initial data by ordinary Cauchy data. (In what follows, we shall refer to their concept of FDs as Dzhrbashyan-Nersesyan FDs.)

For $\beta \in (0, 1)$, we define the Riemann–Liouville fractional integral of order β [14] of a function $z(t)$, $t \geq 0$, by the formula

$$(I^{\beta}_{0+}z)(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{z(s)}{(t-s)^{1-\beta}}ds$$

where $\Gamma(\beta)$ is the Euler Gamma-function. Then the Riemann–Liouville FD of order β has the form

$$(D^{\beta}_{0+}z)(t) = \frac{d}{dt}(I^{1-\beta}_{0+})(t)$$

and the regularized Dzhrbashyan–Nersesyan FD of order β [11, 15] is defined as

$$(D^{(\beta)}z)(t) = (D^{\beta}_{0+}z)(t) - \frac{t^{-\beta}}{\Gamma(1-\beta)}z(+0)$$

With each of the FDs we associate a game problem, as follows. Let the evolution of the conflict-controlled process in the first problem be described by a system of differential equations

$$D^{\beta}\hat{z} = A\hat{z} + \varphi(u, \upsilon), \quad \hat{z} \in R^n, \quad u \in U, \quad \upsilon \in V \tag{2.1}$$

with initial conditions

$$I^{1-\beta}\hat{z}|_{t=0} = \hat{z}_0 \tag{2.2}$$

In the second problem it is described by the system

$$D^{(\beta)}z = Az + \varphi(u, \upsilon), \quad z \in R^n, \quad u \in U, \quad \upsilon \in V \tag{2.3}$$

with initial conditions

$$z|_{t=0} = z_0 \tag{2.4}$$

Some symbols have been omitted from the notation of the FDs in (2.1), (2.3) for simplicity.

Besides the dynamical processes (2.1), (2.3), a terminal set of type (1.2) is given, and the aims of the players in each case are analogous to those described above in the general situation. We merely note that in problems (2.1), (2.2) and (2.3), (2.4) the pursuer chooses as controls measurable functions $u(t) = u(\hat{z}_0, \upsilon_t(\cdot))$ and $u(t) = u(z_0, \upsilon_t(\cdot))$, respectively.

We will now find integral representations of the functions $\hat{z}(t)$ and $z(t)$. To that end, we first define the generalized Mittag–Löffler matrix-valued function

$$E_{\rho}(B; \mu) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(k\rho^{-1} + \mu)}$$

for any positive ρ and complex μ, where B is an arbitrary square matrix of order n with complex-valued elements. The matrix-valued function $E_{\rho}(B; \mu)$ is an entire function of B.

Theorem 2. With the players' controls selected as shown, the solution of problem (2.1), (2.2) is given by the formula

$$\hat{z}(t) = t^{\beta-1}E_{1/\beta}(At^{\beta}; \beta)\hat{z}_0 + z_2(t) \tag{2.5}$$

and the solution of problem (2.3), (2.4) by the formula

$$z(t) = E_{1/\beta}(At^{\beta}; 1)z_0 + z_2(t) \tag{2.6}$$

where

$$z_2(t) = \int_0^t \Omega_{11}(t-\tau)F(\tau)d\tau$$
$$\Omega_{11}(t-\tau) = (t-\tau)^{\beta-1}E_{1/\beta}(A(t-\tau)^{\beta};\beta), \quad F(\tau) = \varphi(u(\tau), v(\tau)) \tag{2.7}$$

Proof. We first note that the function $F(\tau)$ is measurable and essential bounded for $\tau > 0$. Hence the integrals in formulae (2.5) and (2.6) are absolutely convergent.

The proof consists of two parts. In the first part we shall prove that the first terms in formulae (2.5) and (2.6) are solutions of the homogeneous equations satisfying initial conditions (2.2) and (2.4), respectively. In the second, we shall show that the second term in formulae (2.5) and (2.6) is a solution of the inhomogeneous equations (2.1) and (2.3).

That the function $z_2(t)$ satisfies the zero initial conditions follows immediately from the boundedness of the functions $E_{1/\beta}(A(t-\tau)^{\beta};\beta)$ and $F(\tau)$ and from the fact that $\beta > 0$.

Putting

$$\hat{z}_1(t) = t^{\beta-1}E_{1/\beta}(At^{\beta};\beta)\hat{z}_0$$

we proceed to the following calculations

$$(D^{\beta}\hat{z}_1)(t) \equiv D^{\beta}[t^{\beta-1}E_{1/\beta}(At^{\beta};\beta)\hat{z}_0] = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\left(\int_0^t (t-\tau)^{-\beta}\tau^{\beta-1}\sum_{k=0}^{\infty}\frac{A^k\tau^{\beta k}}{\Gamma(\beta k+\beta)}d\tau\right) =$$

$$= \sum_{k=1}^{\infty}\frac{\beta k A^k t^{\beta k-1}}{\Gamma(\beta k+1)}\hat{z}_0 \overset{k=k'+1}{=} At^{\beta-1}\sum_{k'=0}^{\infty}\frac{A^{k'}t^{\beta k'}}{\Gamma(\beta k'+\beta)}\hat{z}_0 = A\hat{z}_1(t)$$

We will show that $\hat{z}_1(t)$ satisfies the initial conditions.We have

$$(I^{1-\beta}\hat{z}_1)(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t\frac{\hat{z}_1(\tau)}{(t-\tau)^{\beta}}d\tau = \sum_{k=0}^{\infty}\frac{A^kt^{\beta k}}{\Gamma(\beta k+1)}\hat{z}_0 \overset{t\to 0}{\to} \hat{z}_0$$

Now consider the function

$$z_1(t) = E_{1/\beta}(At^{\beta};1)z_0 \equiv E_{1/\beta}(At^{\beta})z_0$$

where $E_{1/\beta}(At^{\beta})$ is the Mittag–Löffler matrix-valued function. Then

$$(D^{(\beta)}z_1)(t) = \frac{1}{\Gamma(1-\beta)}\left[\frac{d}{dt}\left(\int_0^t(t-\tau)^{-\beta}\sum_{k=0}^{\infty}\frac{A^k\tau^{\beta k}}{\Gamma(\beta k+1)}d\tau\right) - t^{-\beta}\right]z_0 =$$

$$= A\sum_{k=1}^{\infty}\frac{A^{k-1}t^{\beta(k-1)}}{\Gamma(\beta(k-1)+1)}z_0 = Az_1(t)$$

In addition, the function $z_1(t)$ satisfies the initial conditions (2.4), because

$$\lim_{t\to\infty} z_1(t) = \lim_{t\to 0}\sum_{k=0}^{\infty}\frac{A^kt^{\beta k}}{\Gamma(\beta k+1)}z_0 = z_0$$

We now consider the function $z_2(t)$ defined by formula (2.7) and we will show that is satisfies Eqs (2.1) and (2.3) with zero initial conditions. We have

$$(D^{\beta}z_2)(t) = (D^{(\beta)}z_2)(t) = \frac{1}{\Gamma(1-\beta)}\frac{d\psi}{dt} \tag{2.8}$$

where

$$\psi(t) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\beta k + \beta)} \int_0^t (t-\tau)^{-\beta} \left(\int_0^{\tau} (\tau - s)^{\beta(k+1)-1} F(s) ds \right) d\tau \tag{2.9}$$

Let us investigate the function $\psi(t)$. To do this, we consider the integrals

$$I_k = \int_0^t \int_0^{\tau} (t-\tau)^{-\beta} (\tau - s)^{\beta(k+1)-1} F(s) ds d\tau =$$

$$= \iint_{\Delta_t} (t-\tau)^{-\beta} (\tau - s)^{\beta(k+1)-1} F(s) d\tau ds, \quad \Delta_t = \{(s, \tau): 0 \le s \le \tau \le t\}$$

The last double integral is absolutely convergent, so that by Fubini's theorem the order of integration may be reversed, using Dirichlet's formula. Then

$$I_k = \frac{\Gamma(1-\beta)\Gamma(\beta k + \beta)}{\Gamma(\beta k + 1)} \int_0^t (t-s)^{\beta k} F(s) ds \tag{2.10}$$

It follows from Eqs (2.9) and (2.10) that

$$\psi(t) = \Gamma(1-\beta) \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\beta k + 1)} \int_0^t (t-s)^{\beta k} F(s) ds$$

Since the function $F(t)$ is measurable and bounded, it follows that $\psi(t)$ has a derivative almost everywhere; evaluating the derivative and substituting the result into expression (2.8), we obtain Eq. (2.1) for $\hat{z} = z_2$.

3. FRACTAL GAMES WITH INTEGRAL CONTROL BLOCKS

Along with the conflict-controlled processes (2.1), (2.2) and (2.3), (2.4), we will consider processes that differ from them in having their control blocks in integral form. To be precise: in the case of Riemann–Liouville derivatives we consider the process

$$D^{\beta} \hat{y} = A\hat{y} + \Phi(t), \quad I^{1-\beta} \hat{y}|_{t=0} = \hat{y}_0 \tag{3.1}$$

In the case of regularized Dzhrbashyan–Nersesyan derivatives, the process will be

$$D^{(\beta)} y = Ay + \Phi(t), \quad y|_{t=0} = y_0 \tag{3.2}$$

where

$$\Phi(t) = \int_0^t (t-\tau)^{\gamma-1} \varphi(u(\tau), v(\tau)) d\tau, \quad 0 < \gamma < 1, \quad 0 < \beta < 1 \tag{3.3}$$

Theorem 3. With the players' controls selected as shown, the solution of problem (3.1) is given by the formula

$$\hat{y}(t) = t^{\beta-1} E_{1/\beta}(At^{\beta}; \beta)\hat{y}_0 + y_2(t) \tag{3.4}$$

and the solution of problem (3.2) as given by the formula

$$y(t) = E_{1/\beta}(At^{\beta}; 1) y_0 + y_2(t) \tag{3.5}$$

where

$$y_2(t) = \int_0^t \Omega_{22}(t-\tau)\varphi(u(\tau), v(\tau))d\tau$$
$$\Omega_{22}(t-\tau) = \Gamma(\gamma)(t-\tau)^{\gamma+\beta-1}E_{1/\beta}(A(t-\tau)^{\beta}; \gamma+\beta) \tag{3.6}$$

Taking the proof of Theorem 2 into consideration, it will surface to show that the function $y_2(t)$ is a solution of Eqs (3.1) and (3.2) with zero initial conditions.

Thus, the solutions of game problems with Riemann–Liouville FDs of the type (2.1), (2.2) and (3.1) or Dzhrbashyan–Nersesyan FDs of the type (2.3), (2.4) and (3.2) can be represented by formulae (2.5), (2.6) and (3.4), (3.5), which is a special case of representation (1.1); consequently, the general method presented above may be used to solve each of these problems.

4. THE SOLUTION OF FRACTAL GAMES WITH A SIMPLE MATRIX AND SPHERICAL CONTROL

To illustrate the method, we will consider some special situations in which the computations can be followed through to completion.

In what follows, in the interests of simplicity and universality of the notation, we shall distinguish between the four problems specified above by assigning their parameters index values $i, j = 1, 2$. Thus, a trajectory $z_{11}(t)$ corresponds to process (2.1) with Riemann–Liouville derivatives without integral control block, $z_{12}(t)$ to the same process with integral control block. Again, a trajectory $z_{21}(t)$ corresponds to regularized Dzhrbashyan–Nersesyan derivatives without integral control block, and $z_{22}(t)$ to the same process with integral control block.

We then have four processes

$$z_{ij}(t) = g_{ij}(t) + \int_0^t \Omega_{ij}(t-\tau)\varphi(u(\tau), v(\tau))d\tau, \quad i = 1, 2 \tag{4.1}$$

where

$$g_{11}(t) = G_{11}(t)\hat{z}_0, \quad g_{12}(t) = G_{12}(t)\hat{y}_0 \tag{4.2}$$
$$G_{11}(t) = G_{12}(t) = t^{\beta-1}E_{1/\beta}(At^{\beta}; \beta)$$
$$g_{21}(t) = G_{21}(t)z_0, \quad g_{22}(t) = G_{22}(t)y_0$$
$$G_{21}(t) = G_{22}(t) = E_{1/\beta}(At^{\beta}; 1)$$
$$\Omega_{21}(t-\tau) = \Omega_{11}(t-\tau), \quad \Omega_{12}(t-\tau) = \Omega_{22}(t-\tau)$$

The functions $\Omega_{ii}(t-\tau)$ are defined by the last expressions in formulae (2.7) and (3.6).

Let

$$A = \lambda E, \quad \varphi(u, v) = u - v, \quad M^* = \{0\}, \quad U = aS, \quad a > 1, \quad V = S \tag{4.3}$$

where S is the unit sphere with centre at zero and λ is a number. Then $L = R^n$ and π is the identity operator, given by the identity matrix E. All matrix-valued functions $G_{ij}(t)$ and $\Omega_{ij}(t-\tau)$ have the form

$$G_{ij}(t) = \hat{g}_{ij}(t)E, \quad \Omega_{ij}(t-\tau) = w_{ij}(t-\tau)E; \quad i, j = 1, 2$$

where $\hat{g}_{ij}(t)$ and $w_{ij}(t-\tau)$ are scalar functions. We note merely that the matrix $B = \lambda E$ satisfies the equality

$$E_{\rho}(B; \mu) = E_{\rho}(\lambda; \mu)E$$

where $E_{\rho}(\lambda; \mu)$ is the generalized Mittag–Löffler scalar function [16]. Then

$$W_{ij}(t, \tau, v) = w_{ij}(t-\tau)(aS - v), \quad W_{ij}(t, \tau) = |w_{ij}(t-\tau)|(a-1)S$$

Consequently, Pontryagin's condition holds for $a \geq 1$.

Put $\gamma_{ij}(t, \tau) \equiv 0$. Then

$$\xi_{ij}(t, g_{ij}(t), \gamma_{ij}(t, \cdot)) = g_{ij}(t) = \hat{g}_{ij}(t)z_{ij}^0, \quad z_{ij}^0 \neq 0$$

The quantity

$$\begin{aligned}\alpha_{ij}(t, \tau, \upsilon) &= \sup\{\alpha \geq 0 : \alpha\hat{g}_{ij}(t)z_{ij}^0 \in w_{ij}(t-\tau)(aS - \upsilon)\} = \\ &= (\upsilon_0, q)/\|q\|^2 + \sqrt{(\upsilon_0, q)^2/\|q\|^4 + (a_0^2 - \|\upsilon_0\|^2)/\|q\|^2}\end{aligned} \tag{4.4}$$

is the greatest root of the quadratic equation

$$\|\upsilon_0 - q\alpha\| = a_0$$

with respect to α, where $\upsilon_0 = w_{ij}(t-\tau)\upsilon$, $q = \hat{g}_{ij}(t)z_{ij}^0$, $a_0 = |w_{ij}(t-\tau)|a$.

It should be noted that $\hat{g}_{ij}(t) \neq 0$ right up to the end of the game. The vanishing of this function means that the game can be terminated at the time in accordance with Pontryagin's first direct method [1].

Obviously

$$\min_{\|\upsilon\| \leq 1} \alpha_{ij}(t, \tau, \upsilon) = \frac{(a-1)|w_{ij}(t-\tau)|}{\|\hat{g}_{ij}(t)z_{ij}^0\|}$$

and the minimum is achieved at the element

$$\upsilon_{ij}(t, \tau) = -\operatorname{sign}\{\hat{g}_{ij}(t)w_{ij}(t-\tau)\}\frac{z_{ij}^0}{\|z_{ij}^0\|}$$

Then the termination time of the game is the least root of the equation

$$\int_0^t \frac{(a-1)|w_{ij}(t-\tau)|}{|\hat{g}_{ij}(t)|\|z_{ij}^0\|}d\tau = 1$$

since $w_{ij}(t-\tau)$ are continuous functions; the termination time is defined in each of the cases by the formula

$$T_{ij}(z_{ij}^0, 0) = \min\{t \geq 0 : \Phi_{ij}(t) \geq \xi_{ij}\}$$

where

$$\Phi_{ij}(t) = \int_0^t \frac{|w_{ij}(t-\tau)|}{|\hat{g}_{ij}(t)|}d\tau, \quad \xi_{ij} = \frac{\|z_{ij}^0\|}{a-1} \tag{4.5}$$

The functions $\Phi_{ij}(t)$ take the form

$$\Phi_{i1}(t) = \int_0^t \frac{|G_{11}(\tau)|}{|G_{i1}(t)|}d\tau, \quad \Phi_{i2}(t) = \int_0^t \frac{|\Omega_{12}(\tau)|}{|G_{i1}(t)|}d\tau; \quad i = 1, 2 \tag{4.6}$$

In what follows, asymptotic representations of the generalized scalar Mittag–Löffler function will play an essential role in verifying that the time $T_{ij}(z_{ij}^0, 0)$ at which the game terminates from a given initial state z_{ij}^0 is finite. We shall use well-known formulae [16, p. 134] for such a representation of the function $E_\rho(x; \mu)$ for real x, $\rho > 1/2$ and any μ.

It follows from these formulae that

$$E_\rho(x; \mu) = \chi\rho x^{\rho(1-\mu)}\exp(x^\rho) - \sum_{k=1}^{p} \frac{x^{-k}}{\Gamma(\mu - k\rho^{-1})} + O(|x|^{-(p+1)}) \tag{4.7}$$

where $\chi = 1$ for positive x, and $\chi = 0$ for negative x.

In the example considered it is clearly natural to distinguish two cases: $\lambda > 0$ and $\lambda < 0$.

Let $\lambda > 0$. Then all the generalized Mittag–Löffler functions occurring in the formulae for $\Phi_{ij}(t)$ are positive. We shall use that fact, as well as the formula [16, p. 120]

$$\int_0^x E_\rho(\lambda x^{1/\beta}; \mu)\tau^{\mu-1}d\tau = x^\mu E_\rho(\lambda x^{1/\beta}; \mu+1), \quad \mu > 0, \quad \lambda \in R$$

Then the functions $\Phi_{ij}(t)$ $(i = 1, 2)$ become

$$\Phi_{i1}(t) = \frac{t^\beta E_{1/\beta}(\lambda t^\beta; \beta+1)}{G_{i1}(t)}, \quad \Phi_{i2}(t) = \Gamma(\gamma)\frac{t^{\beta+\gamma}E_{1/\beta}(\lambda t^\beta; \beta+\gamma+1)}{G_{i1}(t)} \tag{4.8}$$

In formula (4.7) $(\chi = 1)$, we put $\rho = 1/\beta, x = \lambda t^\beta$. Note that then, since $\beta \in (0, 1)$, we have $\rho \in (1, \infty)$ and consequently $\rho > 1/2$. Then, using the asymptotic representation

$$E_{1/\beta}(\lambda t^\beta; \mu) = \frac{1}{\beta}(\lambda^{1-\mu/\beta}t)^{1-\mu}\exp(\lambda^{1/\beta}t) + \ldots$$

and formulae (4.8), we find the limits of the functions $\Phi_{ij}(t)$ as $t \to \infty$. As a result, we conclude that for $\lambda > 0$, the time $T_{ij}(z_{ij}^0, 0)$ is finite if the following inequalities hold, respectively:

$$\text{for } i = j = 1{:}\ \lambda^{-1/\beta} > \xi_{11}; \quad \text{for } i = 1, j = 2{:}\ \Gamma(\gamma)\lambda^{-(\gamma+1)/\beta} > \xi_{12};$$

$$\text{for } i = 2, j = 1{:}\ \lambda^{-1} > \xi_{21}; \quad \text{for } i = j = 2{:}\ \Gamma(\gamma)\lambda^{-(\beta+\gamma)/\beta} > \xi_{22}.$$

Now consider the case when $\lambda < 0$. Setting $\chi = 0, \rho = 1/\beta, x = \lambda t^\beta$ in formula (4.7) and using asymptotic representations of the numerators and denominators of the functions $\Phi_{ij}(t)$ defined by formulae (4.8) for $\lambda < 0$, we conclude that

$$\Phi_{ij}(t) \to \infty \quad \text{as} \quad t \to \infty, \quad \forall i, j = 1, 2$$

Thus, the times $T_{ij}(z_{ij}^0, 0)$ defined by formula (4.5) are finite for any z_{ij}^0 $(i, j = 1, 2)$, that is, for the process under consideration, if $\lambda < 0$, we obtain complete conflict controllability [1] for all the problems: (2.1), (2.2); (2.3), (2.4) (3.1) and (3.2).

Let $\lambda = 0$. Then, using formulae (4.8) for the functions $\Phi_{ij}(t)$ $(i, j, = 1, 2)$, as well as the expression (4.5), we obtain exact values for the times at which the games end

$$T_{11}(z_{11}^0, 0) = \beta\xi_{11}, \quad T_{21}(z_{21}^0, 0) = [\Gamma(\beta+1)\xi_{21}]^{1/\beta}$$

$$T_{12}(z_{12}^0, 0) = \left[\frac{\Gamma(\beta+\gamma+1)}{\Gamma(\beta)\Gamma(\gamma)}\xi_{12}\right]^{1/(\gamma+1)}, \quad T_{22}(z_{22}^0, 0) = \left[\frac{\Gamma(\beta+\gamma+1)}{\Gamma(\gamma)}\xi_{22}\right]^{1/(\gamma+\beta)}$$

We wish to dedicate this paper to Academician N. N. Krasovskii's eightieth birthday.

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